

Scientific Report No. 5

Control Theory Group

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2987



THE UNIVERSITY OF TENNESSEE

DEPARTMENT OF ELECTRICAL ENGINEERING

A NEW MEASURE OF POLE SENSITIVITY OF FEEDBACK SYSTEMS

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OTS PRICE

\$ \$

XEROX

MICROFILM

Supported by National Aeronautics and Space Administration
under Grant No. NsG-351.

June 30, 1964

N64-28951

(ACCESSION NUMBER)

31

(PAGES)

CR-58239

(NASA CR OR TMX OR AD NUMBER)

(THRU)

(CODE)

09

(CATEGORY)

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The research reported here was supported by the National
Aeronautics and Space Administration under Research Grant NSG-351.
This support is gratefully acknowledged.

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INTRODUCTION

Engineers have long been using the term "sensitivity" to denote a quantitative measure of the variation of one system parameter due to variation of another system parameter. Different mathematical definitions of sensitivity have been proposed by Bode¹, Truxal², Ur³ and Chang⁴ for different purposes. However, it will be shown that these definitions are not always convenient for practical engineering uses. For instance, does a sensitivity equal to zero really imply the complete insensitiveness of a system? And, does a measure of infinite sensitivity truly mean an infinite change of one system parameter with respect to a finite variation of another parameter?

For engineering purposes, a good definition of sensitivity measure should have the following qualities:

1. significance
2. reliability
3. convenience in use

28957 The object of this paper is to review the various definitions of "sensitivity", and to propose the more satisfactory mathematical definitions of the "pole sensitivity of a feedback system" (hereafter called "pole sensitivity" for simplicity) for engineering applications.

A mathematical definition of pole sensitivity which is more convenient for engineering application will be proposed. The merits of these definitions and their practical reliability will also be demonstrated.

Author

REVIEW OF CURRENT LITERATURE ON SENSITIVITY FUNCTIONS

Bode¹ and Truxal² have defined the sensitivity function as

$$S_x^Q = \frac{d \ln Q}{d \ln x} = - \frac{\frac{dQ}{Q}}{\frac{dx}{x}} \quad (1)$$

which gives the variation of a parameter, or a function, Q due to the variation of another parameter, or function, x . This definition was originally created as a measure of relative variation between the closed-loop gain and one open-loop parameter of a feedback system.

Attempting to use Eq. (1) as a pole sensitivity such that Q may represent any closed-loop pole and x any open-loop parameter gives difficulties under certain conditions. These difficulties will be indicated in the following:

Consider a closed-loop system, Fig. 1, having an open-loop transfer function.

$$G(s) = \frac{Kq(s)}{p(s)} \quad (2)$$

where K is the gain constant and $q(s)$ and $p(s)$ are polynomials in s .

The closed-loop transfer function is

$$T(s) = \frac{C}{R}(s) = \frac{Kq(s)}{p(s) + Kq(s)} = \frac{Kq(s)}{P(s)} \quad (3)$$

Since the dynamic characteristics of a system are largely dependent on the system's pole locations, it is very often desired to know the variation of the closed-loop poles of a feedback system due to the variation of the open-loop gain constant, or time constants (or equivalently, open-loop poles or zeros).

The pole sensitivity, as defined in Eq. (1), for the variation of a closed-loop pole due to the variation of the open-loop gain, open-loop pole or open-loop zero is as follows. Let s_j represent the closed-loop poles and K , α_i , β_i represent the open loop gain, open-loop poles, and open-loop zeros, respectively. And let S_{Kj}^s , $S_{\alpha_i^j}^s$, and $S_{\beta_i^j}^s$ represent the sensitivity functions of s_j with respect to K , α_i , and β_i , respectively. Then (see Appendix I)

$$S_K^{s_j} = - \frac{k_{s_j}}{s_j} \quad (4)$$

$$S_{\alpha_i}^{s_j} = - \frac{k_{s_j}}{s_j} \frac{\alpha_i}{s_j - \alpha_i} = S_K^{s_j} \frac{\alpha_i r_i}{s_j - \alpha_i} \quad (5)$$

$$S_{\beta_i}^{s_j} = \frac{k_{s_j}}{s_j} \frac{\beta_i}{s_j - \beta_i} = -S_K^{s_j} \frac{-\beta_i u_i}{s_j - \beta_i} \quad (6)$$

where

$$k_{s_j} = \left. \frac{Kq(s)}{\frac{d}{ds}P(s)} \right|_{s=s_j} \quad (7)$$

r_i is the multiplicity of the open-loop pole α_i , and u_i is the multiplicity of the open-loop pole β_i . Note that when s_j is a simple pole of the closed-loop function $T(s)$, k_{s_j} is just the residue of $T(s)$ evaluated at $s = s_j$.

When s_j is a multiple pole of $T(s)$, k_{s_j} , as given by Eqs.(7) becomes infinite. Therefore, Eqs. (4), (5) and (6) are not defined. The infinite change of a closed-loop pole due to a differential change of any open-loop parameter is very often a poor approximation for an incremental change of this same open-loop parameter. Therefore, sensitivity measures obtained from Eqs. (4), (5) and (6) often appear to be inconvenient for the practical engineering use. The following example will illustrate this inconvenience.

Consider the feedback system shown in Fig. 1. The open-loop transfer function and the closed-loop transfer function are

$$G_1(s) = \frac{25}{s(s+10)} \quad (8)$$

$$T_1(s) = \frac{C}{R}(s) = \frac{25}{(s+5)^2} \quad (9)$$

respectively. The unit-step response of Eq. (9) is

$$c_1(t) = 1 - e^{-5t} - 5t e^{-5t} \quad (10)$$

Assuming that the open-loop pole is changed 20% from its original value, one then obtains

$$G'(s) = \frac{25}{s(s + 12)} \quad (11)$$

$$T'(s) = \frac{C}{R}(s) = \frac{25}{s^2 + 12s + 25} \quad (12)$$

and the new unit-step response of Eq. (12) is

$$c'(t) = 1 - 1.5e^{-2.683t} + 0.5e^{-9.317t} \quad (13)$$

Eqs. (10) and (13) are plotted in Fig. 2, which indicates a change of about 14% in the rise time response of the closed-loop system due to a 20% change of the open-loop pole. However, in using Eqs. (4), (5) or (6) one finds that the sensitivity of the closed-loop pole at $s = -5$ is infinity, indicating an unlimited change in the closed-loop characteristics due to a very slight change of any open-loop quantity. This shows that the sensitivity functions defined in Eqs. (4), (5) and (6) are indeed inconvenient when the closed-loop transfer function possesses multiple poles.

Furthermore, when a pole s_j of the closed-loop transfer function is at the origin, as is often the case in a sampled-data control system, all sensitivity measures obtained from Eqs. (4), (5) and (6) are again infinity.

Now, consider a sampled-data system as shown in Fig. 3 where

$$G(z) = \frac{1.2z^{-1}(1 - 0.833z^{-1})}{(1 - z^{-1})^2}, \quad (14)$$

and

$$T(z) = \frac{1.2z^{-1}(1 - 0.833z^{-1})}{(1 - 0.8z^{-1})} \quad (15)$$

which has a pole at the origin. The unit-step response as shown in

Fig. 4 is

$$C(z) = 1.2z^{-1} + 1.16z^{-2} + 1.128z^{-3} + 1.1024z^{-4} + 1.082z^{-5} \\ + 1.0656z^{-6} + 1.0525z^{-7} + 1.042z^{-8} + \dots \quad (16)$$

If open-loop gain is changed by 20%, the above equations become

$$G'(z) = \frac{1.44z^{-1}(1 - 0.833z^{-1})}{(1 - z^{-1})^2}, \quad (17)$$

$$T'(z) = \frac{1.44z^{-1}(1 - 0.833z^{-1})}{1 - 0.56z^{-1} - 0.2z^{-2}} \quad (18)$$

and

$$C'(z) = 1.44z^{-1} + 1.0464z^{-2} + 1.11398z^{-3} + 1.07312z^{-4} \\ + 1.06368z^{-5} + 1.05026z^{-6} + 1.01736z^{-7} + 1.00769z^{-8} + \dots \quad (19)$$

Again referring to Fig. 4 and comparing $c(nT)$ with $c'(nT)$, one can easily see that the change of the closed-loop characteristics due to a 20% change of open-loop gain, is not as detrimental as that implied by the infinite sensitivity obtained by using Eq. (4).

Finally, one can easily see from Eqs. (5) and (6) that when the open-loop function has a pole, or zero, at the origin the sensitivity of the closed-loop pole due to a variation of this open-loop pole, or zero, is zero; thus indicating a complete insensitivity of the closed-loop characteristics.

Now again consider Fig. 1 with the following open-loop and closed-loop transfer functions

$$G(s) = \frac{2}{s(s+3)} \quad (20)$$

and

$$T(s) = \frac{2}{(s+1)(s+2)} \quad (21)$$

The unit-step response of the system is then

$$c(t) = 1 - 2e^{-t} + e^{-2t} \quad (22)$$

If the open-loop pole at the origin drifts to $s = -0.2$, Eqs. (20), (21)

and (22) then become,

$$G'(s) = \frac{2}{(s + 0.2)(s + 3)}, \quad (23)$$

$$T'(s) = \frac{2}{s^2 + 3.1s + 2.3} \quad (24)$$

$$\text{and } c'(t) = 0.8696 - 1.1e^{-0.537t} + 0.2305e^{-2.563t} \quad (25)$$

Comparing Eqs. (22) and (25), which are plotted in Fig. 5, indicates a closed-loop gain change of 13% and rise-time change of 78%. These results completely disagree with the zero sensitivity obtained from Eq. (5).

The above examples demonstrate that the form of Bode-Truxal's sensitivity function, Eq. (1), is troublesome when used as a measure of a pole sensitivity under the following conditions:

- (i) When the pole of the closed-loop transfer function, whose sensitivity is desired, has multiplicity greater than one.
- (ii) When the pole of the closed-loop transfer function, whose sensitivity is desired, is at the origin.
- (iii) When the pole, or zero, of the open-loop transfer function, with respect to which the closed-loop sensitivity is desired, is at the origin.

The first two conditions have been shown to yield infinite sensitivity while the third condition yields zero sensitivity.

It should be mentioned that the Bode-Truxal sensitivity definition, Eq. (1) does give correct sensitivity measure under the condition implied by its mathematical form; that is, when the change of x is a differential, dx . However, in nearly every practical system the change of open-loop parameter x is an increment Δx rather than a differential. Therefore, Eq. (1) has practical significance only if

$$\frac{d \ln Q}{d \ln x} = \frac{\Delta \ln Q}{\Delta \ln x} \quad (26)$$

Eq. (26) is not true under the three conditions mentioned above.

Ur³ proposed the following definition of sensitivity

$$S_{\frac{s_j}{x}} = \frac{\frac{ds_j}{dx}}{x}$$

where s_j is the closed-loop pole and x can be any open-loop parameter including gain, pole, or zero. While this definition does not give an infinite sensitivity under condition (ii), it still shows infinite and zero sensitivity under conditions (i) and (iii), respectively.

Chang⁴ has used the definition of sensitivity for a closed-loop pole s_j with respect to a parameter x as

$$S_{\frac{s_j}{x}} = \frac{ds_j}{dx} \quad (27)$$

This definition is better than those of Eqs. (1) and (26), since it yields finite sensitivity under conditions (ii) and (iii). But under conditions (i), when the multiplicity of the closed-loop pole is greater than one, the partial derivative of Eq. (27) does not exist.

In the following section a definition of sensitivity which is more convenient for engineering applications, will be proposed and its effectiveness will be demonstrated.

PROPOSED DEFINITIONS OF POLE SENSITIVITY

In general, the relation between the variation of a closed-loop pole Δs_j due to a variation of an open-loop parameter Δx can be expressed in terms of a Taylor series as

$$\begin{aligned}\Delta x &= a_1 \Delta s_j + a_2 (\Delta s_j)^2 + a_3 (\Delta s_j)^3 + \dots \\ &= \sum_{r=1}^{\infty} a_r (\Delta s_j)^r\end{aligned}\quad (28)$$

where $a_r = \frac{1}{r!} \frac{d^r x}{ds_j^r}$

When the closed-loop pole is of multiplicity m , the first $m-1$ coefficients of Eq. (28) vanish, (See Appendix II) i.e.

$$a_1 = a_2 = \dots = a_{m-1} = 0. \quad (29)$$

Under this condition Eq. (28) becomes

$$\Delta x = \sum_{r=m}^{\infty} a_r (\Delta s_j)^r. \quad (30)$$

The single term approximation of this equation is

$$\Delta x = a_m (\Delta s_j)^m \quad (31)$$

where $a_m = \frac{1}{m!} \frac{d^m x}{ds_j^m}$

When the parameter x is the open-loop gain K , the division of Eq. (31) by K yields

$$\frac{\Delta K}{K} = \frac{a_m}{K} (\Delta s_j)^m \quad (32)$$

due to the availability of the coefficients of Eq. (32) the sensitivity of the closed-loop pole s_j with respect to the open-loop gain K may be defined as

$$S_K^{s_j} = \frac{(\Delta s_j)^m}{\frac{\Delta K}{K}}. \quad (33)$$

Using Eqs. (31) and (32), Eq. (34) then becomes

$$S_K^s = \frac{m! K}{\frac{d^m K}{ds_j^m}} \quad (34)$$

Eq. (33) or (34), gives a measure of closed-loop pole deviation due to the fractional change of open-loop gain. For ΔK small Eq. (33) approaches its differential form,

$$S_K^s = \frac{(ds_j)^m}{d(\ln K)} \text{ pole}^m / \text{neper} \quad (35)$$

where the gain variation $d \ln K$ is in nepers. If the unit of gain variation in db is desired, Eq. (35) becomes

$$S_K^s = 0.115 \frac{(ds_j)^m}{d(\ln K)} = \frac{0.115 m! K}{\frac{d^m K}{ds_j^m}} \quad (36)$$

When the parameter x is the open-loop pole or zero, the sensitivity of the closed-loop pole with respect to x is defined as

$$S_x^s = \frac{(\Delta s_j)^m}{\Delta x} \quad (37)$$

Using Eq. (31), Eq. (37) becomes

$$S_x^s = \frac{m!}{\frac{d^m x}{ds_j^m}} \quad (38)$$

Eq. (37) or (38) gives a measure of the closed-loop pole variation due to the variation of the open-loop pole or zero.

In essence, the proposed sensitivities, Eqs. (34) and (38), are defined to be the reciprocal of the first non-zero coefficient of the Taylor series expansion in Eq. (28).

Examining the form of Eqs. (34) and (38), one can easily see that these sensitivity functions do not suffer any difficulty under conditions (i), (ii) and (iii) described in previous section.

EVALUATION OF SENSITIVITIES

If Eqs. (34) and (38) are used to evaluate the sensitivities S_K^s and S_x^s respectively, it is necessary to obtain the m-th derivatives by repeated differentiation, a process which is very cumbersome for any second or higher order system. Following the approach of Papoulis⁵, who suggested a method of determining the zeros of the impedance function due to incremental variations in the network element, the variation of the closed-loop pole due to the variation of the open-loop parameter can be obtained in a much easier manner.

The characteristic equation of the closed-loop system shown in Fig. 1 is

$$p(s) + Kq(s) = 0, \quad (39)$$

where $p(s)$ and $q(s)$ are respectively the denominator and numerator polynomials of the open-loop function. The values of various sensitivity functions defined in the third section are given by the following formulas:

$$S_K^s = \frac{(\Delta s_j)^m}{\frac{\Delta K}{K}} = - \left[(s - s_j)^m T(s) \right]_{s \rightarrow s_j} \quad (40)$$

where $T(s) = \frac{G(s)}{1 + G(s)}$.

$$S_{\alpha_i}^s = \frac{(\Delta s_j)^m}{\Delta \alpha_i} = \left[\frac{(s - s_j)^m p(s)}{P(s) + Kq(s) (s - \alpha_i)} \right]_{s \rightarrow s_j} \quad (41)$$

where α_i is the pole of open-loop function whose variation causes the closed-loop s_j to vary.

$$S_{\beta_i}^s = \left[\frac{(s - s_j)^m}{(s - \beta_j)} T(s) \right]_{s \rightarrow s_j} \quad (42)$$

where β_i is the zero of open-loop function whose variation causes the closed-loop pole to vary.

The derivation of Eqs. (40) through (42) are included in Appendix III.

The sensitivities evaluated using these formulas are indeed consistent with the proposed definitions in the third section. This is proved in Appendix IV.

Although the derivation here is for unit feedback system, the results can be applied to non-unity feedback system with the aid of block diagram transformation.

EXAMPLES

Example 1. Referring to Fig. 1, let $G(s) = \frac{25}{s(s+10)}$. Then $T(s) = \frac{25}{(s+5)^2}$ which has a double pole at $s_j = -5$. Find the sensitivity $S_K^{s_j}$.

From Eq. (40), the sensitivity $S_K^{s_j}$ can be found as

$$\begin{aligned} S_K^{s_j} &= \frac{(\Delta s_j)^m}{\frac{\Delta K}{K}} = - \left[(s - s_j)^m T(s) \right]_{s \rightarrow s_j} \\ &= - \left[(s+5)^2 \frac{25}{(s+5)^2} \right]_{s \rightarrow -5} = -25. \quad (43) \end{aligned}$$

Let the gain of $G(s)$ change -4%, i. e. $\Delta K = -1$, then the variation of the closed-loop pole can be obtained from

$$\begin{aligned} (\Delta s_j)^m &= (\Delta s_j)^2 = \frac{\Delta K}{K} S_K^{s_j} \\ &= (0.04)(25) = 1 \\ \Delta s_j &= \pm 1 \quad (44) \end{aligned}$$

To check the reliability of Eq. (43), one easily finds the new closed-loop transfer function

$$T'(s) = \frac{24}{(s+6)(s+4)}.$$

which shows the variation of the closed-loop pole is indeed $\Delta s_j = \pm 1$.

Example 2. Referring to Fig. 3, let

$$G(z) = \frac{1.2z^{-1}(1 - 0.833z^{-1})}{(1 - z^{-1})^2}$$

then

$$T(z) = \frac{1.2(z - 0.833)}{z(z - 0.8)}$$

which has a pole at the origin. Considering a 20% change of open-loop

from $K = 1.2$ to $K + \Delta K = 1.44$ find S_K^j with $s_j = 0$ and 0.8 .

Again use Eq. (40),

$$S_K^{s_j=0} = \left[z \frac{1.2(z - 0.833)}{z(z - 0.8)} \right]_{z \rightarrow 0} = 1.25, \quad (45)$$

and

$$S_K^{s_j=0.8} = \left[(z - 0.8) \frac{1.2(z - 0.833)}{z(z - 0.8)} \right]_{z \rightarrow 0.8} = -0.0495. \quad (46)$$

The variation of the closed-loop poles are obtained from

$$(\Delta s_j)^m = \frac{\Delta K}{K} S_K^j$$

giving,

$$\text{at } s_j = 0, \quad \Delta s_j = (0.20)(1.25) = 0.250 \quad (47)$$

$$\text{at } s_j = 0.8, \quad \Delta s_j = (0.20)(-0.0495) = 0.0099 \quad (48)$$

To check the results, one finds the new closed-loop transfer function

$$T'(z) = \frac{1.44(z - 0.833)}{(z + 0.249)(z - 0.809)}$$

indicating

$$\text{at } s_j = 0, \quad \Delta s_j = 0.249 \quad (49)$$

$$\text{at } s_j = 0.8, \quad \Delta s_j = -0.009 \quad (50)$$

Eqs. (45) and (50) confirms the results of Eqs. (47) and (48) which were obtained by using the new sensitivity function.

Example 3. Using again Fig. 1, let $G(s) = \frac{10}{s(s + 11)}$ which has a pole at the origin. The $T(s) = \frac{10}{(s + 1)(s + 10)}$. Find the sensitivity of the closed-loop poles with respect to open-loop pole at $\alpha_i = 0$.

Using Eq. (41)

$$S_{\alpha_i}^{s_j=-1} = \left[\frac{(s + 1)s(s + 11)}{(s + 1)(s + 10)s} \right]_{s = -1} = 1.11 \quad (51)$$

$$S_{\alpha_i}^{s_j=-10} = \left[\frac{(s+10)s(s+11)}{(s+1)(s+10)s} \right]_{s=-10} = -0.111 \quad (52)$$

Assuming a change of open-loop pole $\Delta\alpha_i = -0.2$, the variations of the closed loop pole are obtained from, by Eq. (37),

$$(\Delta s_j)^m = \Delta\alpha_i S_{\alpha_i}^{s_j}$$

giving

$$\Delta s_j = (-0.2)(1.11) = -0.222 \quad \text{for } s_j = -1, \quad (53)$$

$$\Delta s_j = (-0.2)(-0.111) = +0.0222 \quad \text{for } s_j = -10 \quad (54)$$

To check the results, one first obtains the new closed-loop transfer function

$$T'(s) = \frac{10}{(s+1.222)(s+9.98)}$$

The exact variations of the closed loop poles are therefore

$$\Delta s_j = -0.222 \quad \text{for } s_j = -1, \quad (55)$$

$$\Delta s_j = +0.02 \quad \text{for } s_j = -10. \quad (56)$$

which compare very well with Eqs. (53) and (54).

The following table summarizes and compares the results of the above examples to the results one would have if the conventional sensitivity function, Eq. (1), was used. One, therefore, sees the reliability of the proposed definitions of pole sensitivity.

Example	Property	Varying Parameter	Variation of closed-loop pole obtained from:		
			Proposed sensitivity definition	Conventional sensitivity definition	Exact Value
1	Closed-loop pole has multiplicity 2	Open loop gain	+1	∞	+1
2	Closed-loop pole at the origin	Open loop gain	0.250	∞	0.249
			0.0099	∞	-0.009
3	Open-loop pole at the origin	Open-loop pole	-0.222	0	-0.222
			+0.0222	0	+0.02

Table I. Results of Examples

CONCLUSION

In this paper the inconvenience of the conventional definitions of pole sensitivity have been examined and discussed. New definitions of pole sensitivity have been proposed. In essence, the proposed sensitivities are defined to be reciprocal of the first non-zero coefficient of the power series, Eq. (28). Convenient method of evaluating the new sensitivity function was explored. Examples were given to illustrate the merits and the reliability of new definitions.

Further research to apply the proposed definition of pole sensitivity to network theory is under way.

ACKNOWLEDGMENT

The author thanks Dr. N. K. Sinha for his comments, Mr. C. F. Chen for helping in preparation of the text, and Mrs. Nancy Fields for typing the manuscript.

This research was supported by the National Aeronautics and Space Administration under Research Grant NSG-351. The support is gratefully acknowledged.

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APPENDIX I. DERIVATION OF EQS. (4) THROUGH (7)

Let the open-loop transfer function shown in Fig. 1 be

$G(s) = \frac{Kq(s)}{p(s)}$, where K is the gain of the open-loop function. Thus the characteristic equation is:

$$p(s) + Kq(s) = 0. \quad (57)$$

(i) When the open-loop gain K is the variable parameter, the differentiation of Eq. (57) with respect to K , gives

$$\frac{ds}{dK} = - \frac{q(s)}{p'(s) + Kq'(s)} \quad (58)$$

The sensitivity is then

$$S_K^s = \frac{d \ln s}{d \ln K} = - \frac{Kq(s)}{p'(s) + Kq'(s)} \frac{1}{s}. \quad (59)$$

At a particular closed-loop pole s_j , the pole sensitivity with respect to the gain K is

$$S_K^{s_j} = \frac{\frac{ds_j}{s_j}}{\frac{dK}{K}} = - \left[\frac{Kq(s)}{p'(s) + Kq'(s)} \frac{1}{s} \right]_{s \rightarrow s_j} = - \frac{ks_j}{s_j}, \quad (60)$$

$$\text{where } ks_j = \left. \frac{Kq(s)}{p'(s) + Kq'(s)} \right|_{s = s_j}$$

Eq. (60) is the same as Eq. (4).

(ii) When the open-loop pole α_i with multiplicity r , is the variable parameter, Eq. (60) becomes

$$p_i(s)(s - \alpha_i)^r + Kq(s) = 0, \quad (61)$$

$$\text{where } p_i(s) = \frac{p(s)}{(s - \alpha_i)^r}$$

Differentiating Eq. (61) with respect to α_i ,

$$p'(s) + Kq'(s) \frac{ds}{d\alpha_i} = r_i p_i(s)(s - \alpha_i)^{r_i-1} d\alpha_i$$

$$\text{then, } \frac{ds}{d\alpha_i} = \frac{p(s)}{p'(s) + Kq'(s)} \frac{r_i}{s - \alpha_i} = \frac{-Kq(s)}{p'(s) + Kq'(s)} \frac{r_i}{s - \alpha_i} \quad (62)$$

The sensitivity of the closed-loop pole s_j with respect to the open-loop pole α_i is

$$S_{\alpha_i}^{s_j} = \frac{\frac{ds_j}{s_j}}{\frac{d\alpha_i}{\alpha_i}} = - \left[\frac{1}{s} \frac{Kq(s)}{p'(s) + Kq'(s)} \frac{i^r i}{s - \alpha_i} \right]_{s = s_j}$$

$$= - \left[\frac{ks_j}{s_j} \right] \frac{i^r i}{s_j - \alpha_i} = S_K^{s_j} \frac{i^r i}{s_j - \alpha_i} . \quad (63)$$

This is Equation (5)

(iii) When the open-loop zero β_i , with multiplicity u_i is the variable parameter, Eq. (57) becomes

$$p(s) + Kq_i(s)(s - \beta_i)^{u_i} = 0, \quad (64)$$

$$\text{where } q_i(s) = \frac{q(s)}{(s - \beta_i)^{u_i}}$$

Differentiating Eq. (64) with respect to β_i ,

$$\frac{ds}{d\beta_i} = \frac{Kq_i(s)}{p'(s) + Kq'(s)} = \frac{Kq(s)}{p'(s) + Kq'(s)} \frac{u_i}{s - \beta_i} .$$

The sensitivity of the closed-loop pole s_j with respect to the open-loop zero β_i is then

$$S_{\beta_i}^{s_j} = \frac{\frac{ds_j}{s_j}}{\frac{d\beta_i}{\beta_i}} = \left[\frac{1}{s} \frac{Kq(s)}{p'(s) + Kq'(s)} \frac{\beta_i u_i}{s - \beta_i} \right]_{s = s_j}$$

$$= \left[\frac{ks_j}{s_j} \right] \frac{\beta_i u_i}{s_j - \beta_i} = -S_K^{s_j} \frac{\beta_i u_i}{s - \beta_i} . \quad (65)$$

This is Eq. (6).

APPENDIX II. PROOF OF EQ. (29)

In Fig. 1, the characteristic equation of the closed loop system is

$$D(s, x) = p + Kq = 0 \quad (66)$$

where x is a system parameter and s is Laplace transform variable. If D has a root s_1 of multiplicity m , one can write

$$D(s) = (s - s_1)^m D_1(s). \quad (67)$$

Note that

$$\left[\frac{d^k}{ds^k} D(s, x) \right]_{s=s_1} \begin{cases} = 0 & \text{for } k = 1, \dots, m-1 \\ \neq 0 & \text{for } k \geq m \end{cases} \quad (68)$$

Taking the differential of Eq. (67),

$$D_s + D_x \frac{dx}{ds} = 0$$

or

$$\frac{dx}{ds} = - \frac{D_s}{D_x} \quad (69)$$

where D_s and D_x are the derivatives of D with respect to s and x , respectively. By Eq. (68), one finds

$$\left[\frac{dx}{ds} \right]_{s=s_1} = - \left[\frac{D_s}{D_x} \right]_{s=s_1} = 0 \quad (70)$$

By repeatedly differentiating Eq. (69) and using Eq. (68), one easily find

$$\left[\frac{d^k x}{ds^k} \right]_{s=s_1} \begin{cases} = 0 & \text{for } k = 1, 2, \dots, m-1, \\ \neq 0 & \text{for } k = m \end{cases} \quad (71)$$

Therefore,

$$a_k = \frac{1}{k!} \left[\frac{d^k x}{ds^k} \right]_{s=s_1} \begin{cases} = 0 & \text{for } k = 1, 2, \dots, m-1 \\ = 0 & \text{for } k \leq m. \end{cases} \quad (72)$$

APPENDIX III. DERIVATION OF EQS. (40) THROUGH (42)

In Fig. 1, the open-loop transfer function is

$$G(s) = \frac{Kq(s)}{p(s)}. \quad (73)$$

The closed-loop transfer function is

$$T(s) = \frac{C(s)}{R(s)} = \frac{Kq(s)}{p(s) + Kq(s)}, \quad (74)$$

which has a pole s_j of multiplicity m .

First find the sensitivity of the closed-loop pole s_j with respect to open-loop gain K . For a normal value of gain K , the closed-loop pole s_j satisfies the equation.

$$p(s_j) + Kq(s_j) = 0. \quad (75)$$

If the gain K changes to $K + \Delta K$, the new closed-loop pole s'_j must satisfy the equation

$$p(s'_j) + (K + \Delta K)q(s'_j) = 0$$

$$p(s'_j) + Kq(s'_j) = -q(s'_j)\Delta K. \quad (76)$$

Multiplying both sides of Eq. (76) by $\frac{(s'_j - s_j)^m}{K}$ and rearranging the terms,

$$(s'_j - s_j)^m = \frac{-(s'_j - s_j)^m}{p(s'_j) + Kq(s'_j)} Kq(s'_j) \left(\frac{\Delta K}{K} \right) \quad (77)$$

Since $p + Kq$ contains $(s - s_j)^m$ as its factor,

$$s'_j \lim_{s'_j \rightarrow s_j} \frac{(s'_j - s_j)^m Kq(s'_j)}{p(s'_j) + Kq(s'_j)} \neq 0.$$

Therefore for small $\Delta s_j = s'_j - s_j$

$$(\Delta s_j)^m = \left[\frac{-(s - s_j)^m Kq(s)}{p(s) + Kq(s)} \right]_{s \rightarrow s_j} \left(\frac{\Delta K}{K} \right)$$

$$= - \left[(s - s_j)^m T(s) \right]_{s \rightarrow s_j} \left(\frac{\Delta K}{K} \right) \quad (78)$$

where s has been used in place of s'_j . Eq. (78) is the same as Eq. (40).

Next find the sensitivity of the closed-loop pole s_j with respect to the open-loop pole α_i . Write Eq. (75) as

$$(s_j - \alpha_i) p_i(s_j) + Kq(s_j) = 0, \quad (79)$$

where α_i is the open-loop pole and

$$p_i(s) = \frac{p(s)}{s - \alpha_i}. \quad (80)$$

For a change of α_i by $\Delta\alpha_i$, the new closed-loop pole must satisfy

$$(s'_j - \alpha_i - \Delta\alpha_i) p_i(s'_j) + Kq(s'_j) = 0$$

or

$$(s'_j - \alpha_i) p_i(s'_j) + Kq(s'_j) = p_i(s'_j) \Delta\alpha_i. \quad (81)$$

Multiplying this Eq. (81) by $(s'_j - s_j)^m$,

$$\begin{aligned} (s'_j - s_j)^m &= \frac{(s'_j - s_j)^m p_i(s'_j)}{(s'_j - \alpha_i) p_i(s'_j) + Kq(s'_j)} \Delta\alpha_i \\ &= \frac{(s'_j - s_j)^m p_i(s'_j)}{p(s'_j) + Kq(s'_j)} \Delta\alpha_i \\ &= \frac{(s - s_j)^m p(s)}{(s'_j - \alpha_i) p(s'_j) + Kq(s'_j)} \Delta\alpha_i \end{aligned} \quad (82)$$

By the same method used to obtain Eq. (78),

$$(\Delta s_j)^m = \left[\frac{(s - s_j)^m p(s)}{(s - \alpha_i) p(s) + Kq(s)} \right]_{s \rightarrow s_j} \Delta\alpha_i \quad (83)$$

Eq. (83) is the same as Eq. (41)

Finally, the sensitivity of the closed-loop pole s_j with respect to the open-loop zero β_i can be derived in a similar manner.

APPENDIX IV. THE EQUIVALENCE OF EQS. (33) AND (40)

In Fig. 1 the characteristic equation of the closed-loop system is

$$p(s) + Kq(s) = 0. \quad (84)$$

If K is the variable parameter then it may be written as

$$K = K_0 + \frac{dK}{ds_j} \Delta s_j + \frac{1}{2!} \frac{d^2 K}{ds_j^2} (\Delta s_j)^2 + \dots \quad (85)$$

Then by Eqs. (32) and (33)

$$S_K^{s_j} = \frac{(\Delta s_j)^m}{\frac{d^m K}{ds_j^m}} = \left[\frac{\frac{m! K}{ds_j^m}}{\frac{d^m K}{ds_j^m}} \right]_{s \rightarrow s_j} \quad (86)$$

When $m=1$, using Eqs. (84) and (86),

$$\begin{aligned} S_K^{s_j} &= \left[\frac{K}{\frac{dK}{ds}} \right]_{s=s_j} = - \left[\frac{Kq(s)}{p'(s) + Kq'(s)} \right]_{s \rightarrow s_j} \\ &= - \left[(s - s_j) T(s) \right]_{s=s_j} \end{aligned} \quad (87)$$

$$\text{where} \quad T(s) = \frac{C(s)}{R(s)} = \frac{Kq(s)}{p(s) + Kq(s)}. \quad (88)$$

when $m=2$, Eqs (84) and (86) give

$$\begin{aligned} S_K^{s_j} &= \left[\frac{\frac{2! K}{ds^2}}{\frac{d^2 K}{ds^2}} \right]_{s \rightarrow s_j} = - \left[\frac{2! Kq(s)}{p''(s) + Kq''(s)} \right]_{s \rightarrow s_j} \\ &= - \left[(s - s_j)^2 T(s) \right]_{s \rightarrow s_j} \end{aligned} \quad (89)$$

In general, therefore, when $m=m$, one has Eqs. (84) and (86)

$$\begin{aligned} S_K^{s_j} &= \left[\frac{\frac{m! K}{ds^m}}{\frac{d^m K}{ds^m}} \right]_{s=s_j} = - \left[\frac{m! Kq(s)}{p^{(m)}(s) + Kq^{(m)}(s)} \right]_{s=s_j} \\ &= - \left[(s - s_j)^m T(s) \right]_{s=s_j} \end{aligned} \quad (90)$$

This is the same as Eq. (40)

In a similar manner, one can show the equivalence of Eqs. (37) and (41), (42).

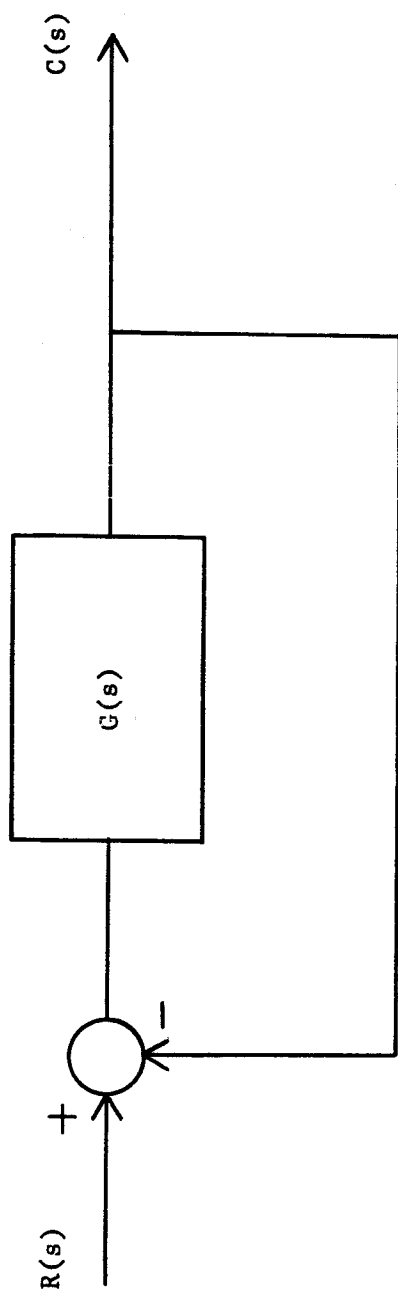


Figure 1. Control System with Unity Feedback

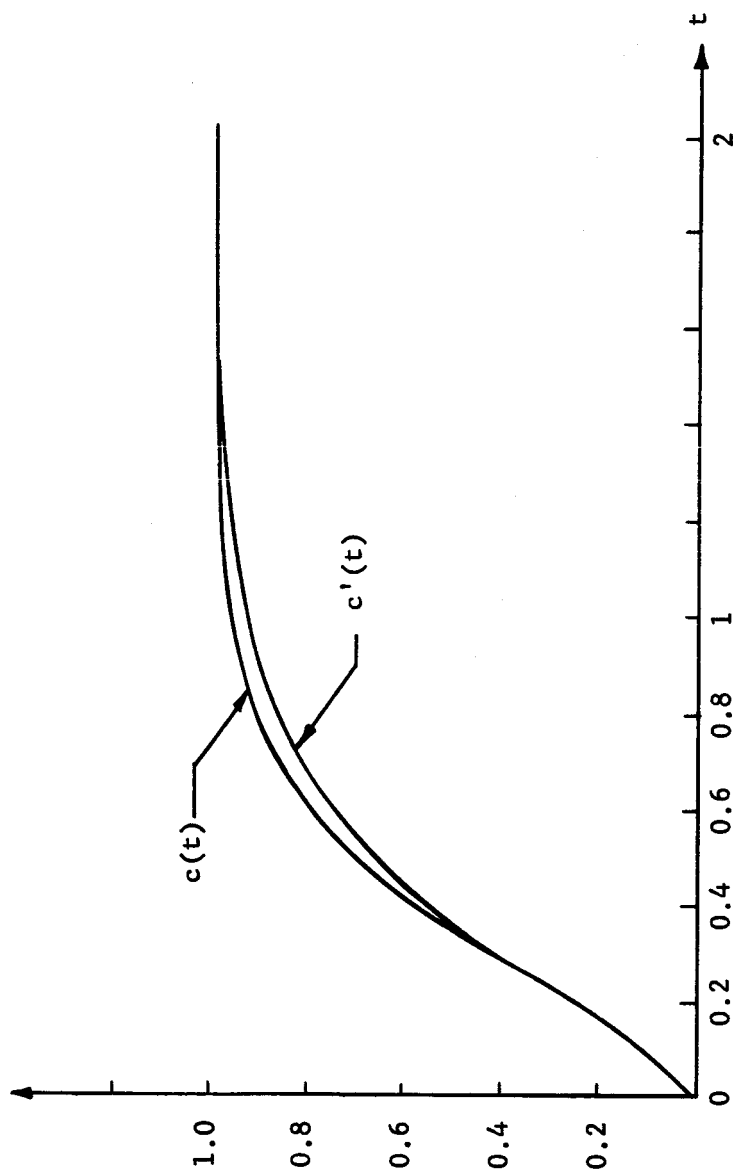


Figure 2. Unit Step Responses Expressed by Eqs. (10, (13)

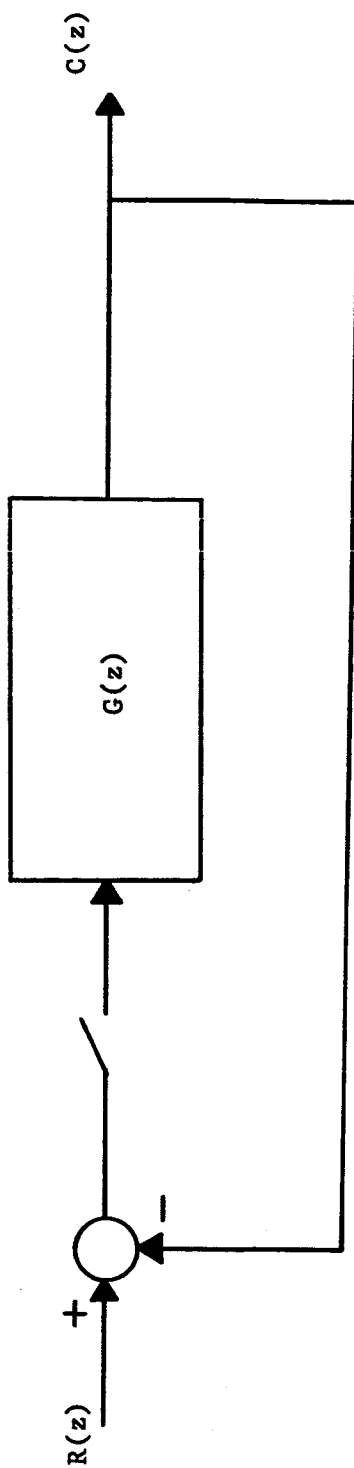


Figure 3. Sampled-Data Control System with Unity Feedback

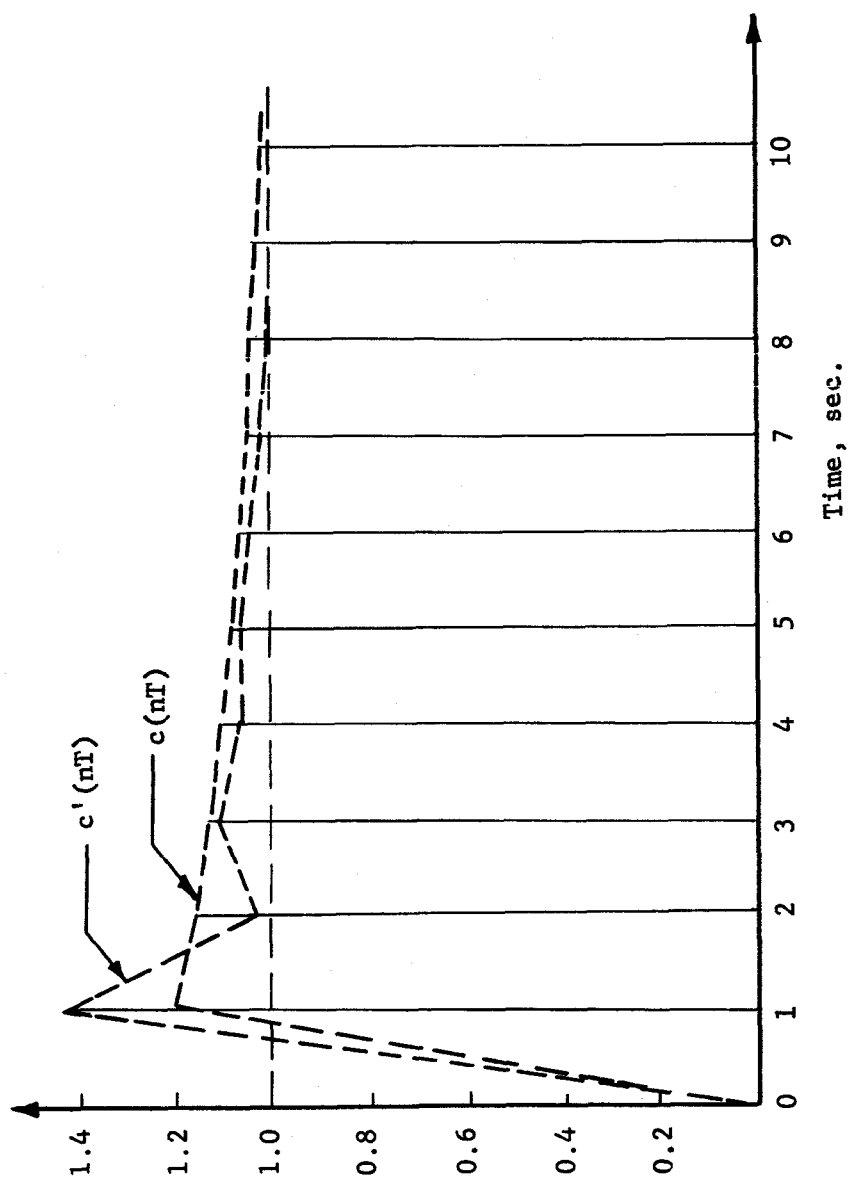


Figure 4. Unit Step Expressed by Eqs. (16) and (19)

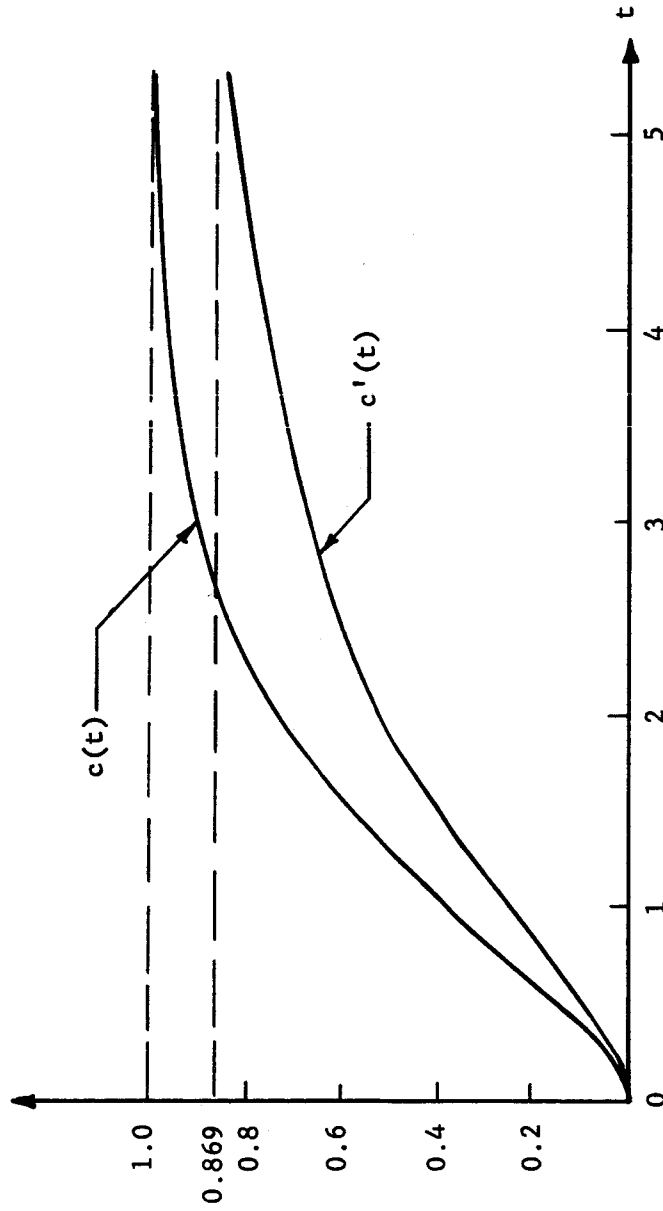


Figure 5. Unit Step Responses Expressed by Eqs. (22) and (25)